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The spectral dimension of 2D quantum gravity

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Abstract: We show that the spectral dimension d_s of two-dimensional quantum gravity coupled to Gaussian fields is two for all values of the central charge $c \leq 1$. The same arguments provide a simple proof of the known result $d_s = 4/3$ for branched polymers.

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Contents

1 Introduction

One of the main objectives of two-dimensional quantum gravity is to gain an understanding of the fractal structure of space-time. The fractal structure reflects true quantum phenomena. In the case of pure two-dimensional quantum gravity, i.e. when the central charge c of conformal matter coupled to gravity is equal to zero, we know that the *intrinsic* Hausdorff dimension is four (not two as for smooth, fixed geometries) [1, 2], and for conformal matter with $c = -2$ it is $3.56\cdots$ [3, 4, 5]. These results can be obtained by calculating the average volume $V(R)$ of balls of geodesic radius R. However, there are other definitions of the fractal structure of space-time which might reflect other aspects of the average space-time one encounters in two-dimensional quantum gravity. One such measure of the fractal properties of space-time is the so-called *spectral* dimension, which can be defined in an reparameterization invariant way and consequently makes sense in quantum gravity.

The most intuitive definition of the spectral dimension is based on the diffusion equation on a (compact) manifold with metric g_{ab}. Let Δ_g denote the Laplace-Beltrami operator corresponding to g_{ab}. The probability distribution $K(\xi, \xi'; T)$ of diffusion is related to the massless scalar propagator $(-\Delta_g)^{-1}$ by [1]

$$\left\langle \xi' \middle| (-\Delta_g)^{-1} \middle| \xi \right\rangle' = \int_0^\infty dT\ K'(\xi, \xi'; T)\,. \tag{1.1}$$

In particular, the average return probability distribution at time T has the following small T behavior:

$$RP'_g(T) \equiv \frac{1}{V_g} \int \mathrm{d}^d\xi \sqrt{g}\ K'(\xi, \xi; T) \sim \frac{1}{T^{d/2}}(1 + O(T))\,, \tag{1.2}$$

where V_g denotes the volume of the compact manifold with metric g. The important point, in relation to quantum gravity, is that $RP'_g(T)$ is invariant under reparameterization. Thus

[1] Since we consider compact manifolds the Laplace-Beltrami operator Δ_g has zero modes. Eq. (1.1) should be understood with these zero modes projected out. This is indicated with a prime.

the quantum average over geometries can be defined:

$$RP'_V(T) \equiv \frac{1}{Z_V} \int \mathcal{D}[g]_V \, \mathrm{e}^{-S_{\mathrm{eff}}([g])} RP'_g(T) \,, \tag{1.3}$$

where Z_V denotes the partition function of quantum gravity for fixed space-time volume V (see (2.1) for more details about Z_V and $\mathcal{D}[g]_V$), and $S_{\mathrm{eff}}([g])$ denotes the effective action of quantum gravity after the integration over possible matter fields. The *spectral* dimension d_s in quantum gravity is now defined by the small T behavior of the functional average $RP'_V(T)$

$$RP'_V(T) \sim \frac{1}{T^{d_s/2}}(1 + O(T)) \,. \tag{1.4}$$

The $O(T)$ term in (1.2) has a well known asymptotic expansion in powers of T, where the coefficient of T^r is an integral over certain powers and certain contractions of the curvature tensor. This asymptotic expansion breaks down when $T \sim V^{2/d}$ at which point the exponential decay in T of the heat kernel K takes over. If we average over all geometries as in (1.3), it is natural to expect that the only invariant left will be the volume V which is kept fixed. Thus we expect that we can write

$$RP'_V(T) = \frac{1}{T^{d_s/2}} F\Big(\frac{T}{V^{2/d_s}}\Big) \,, \tag{1.5}$$

where $F(0) > 0$ and $F(x)$ falls off exponentially fast for $x \to \infty$.

For fixed manifold of dimension d and a given smooth geometry $[g]$ we have $d = d_s$ by definition. The functional average can *a priori* change this, i.e. the dimension of T can become anomalous. A well known example of similar nature can be found for the ordinary free particle. In the path integral representation of the free particle, any smooth path of course has fractal dimension equal to one. Nevertheless the short distance properties of the free particle reflects that the generic path contributing to the path integral has fractal dimension (the *extrinsic* Hausdorff dimension – in the target space R^D) $D_H = 2$ with probability one. In the same way the functional integral over geometries might change d_s from the "naive" value d. In two-dimensional quantum gravity it is known, as mentioned above, that the intrinsic Hausdorff dimension *is* different from $d = 2$ and the generic geometry is in this sense fractal, with probability one. When one considers diffusion on fixed fractal structures (often embedded in R^D) it is well known that d_s can be different from both D and the fractal dimension d_h of the structure. If δ denotes the so-called anomalous gap exponent, defined by the relation between the diffusion time T and the average spread of diffusion on the fractal structure, but measured in R^D:

$$\Big\langle r^2(T) \Big\rangle \sim T^{2/\delta} \,, \tag{1.6}$$

then the relation between the fractal dimension (*intrinsic* Hausdorff dimension) d_h of the structure, the spectral dimension of the diffusion and the gap exponent is:

$$d_s = \frac{2d_h}{\delta} \,. \tag{1.7}$$

If δ is not anomalous, i.e. $\delta = 2$ as for diffusion on a smooth geometry, we have $d_s = d_h$, which is the analogue of $d_s = d$ for fixed smooth geometries. However, in general $\delta \neq 2$ (for a review of diffusion on fractal structure, see e.g. [6]).

In principle the above definitions apply for any theory of quantum gravity (see for instance [7, 8, 9] for definitions in the case of four dimensional simplicial quantum gravity). However, only for two-dimensional quantum gravity we have presently a well defined theory which allows us to make detailed calculations. In the following we will not consider higher dimensional quantum gravity.

2 Spectral dimension for 2d quantum gravity

We will derive a simple relation between the spectral dimension and the extrinsic Hausdorff dimension for dynamical self-similar systems like two-dimensional quantum gravity, branched polymers etc. Since the extrinsic Hausdorff dimension is known for these systems it will allow a determination of the spectral dimension. As a typical example of such models (and maybe the most interesting), we consider two-dimensional quantum gravity. The partition function for two-dimensional quantum gravity coupled to D Gaussian fields X_μ is given by

$$Z_V = \int \mathcal{D}[g]_V \mathcal{D}[X_\mu]_{\rm cm}\, \mathrm{e}^{-\int \mathrm{d}^2\xi \sqrt{g}\, g^{ab}\partial_a X_\mu \partial_b X_\mu}\,, \tag{2.1}$$

where $\int \mathcal{D}[g]_V$ denotes the integration over *geometries*, i.e. equivalence classes of metrics on the two-dimensional manifold of fixed space-time volume V of the manifold, while $\mathcal{D}[X_\mu]_{\rm cm}$ denotes the functional integration over the D Gaussian fields X_μ, but with the center of mass fixed (to zero). The extrinsic Hausdorff dimension D_H is usually defined as

$$\left\langle X^2 \right\rangle_V \sim V^{2/D_H} \quad \text{for} \quad V \to \infty, \tag{2.2}$$

where

$$\left\langle X^2 \right\rangle_V \equiv \frac{1}{Z_V} \int \mathcal{D}[g]_V \mathcal{D}[X_\mu]_{\rm cm}\, \mathrm{e}^{-\int \mathrm{d}^2\xi \sqrt{g}\, g^{ab}\partial_a X_\mu \partial_b X_\mu}\, \frac{1}{DV} \int \mathrm{d}^2\xi \sqrt{g}\, X_\mu^2(\xi)\,. \tag{2.3}$$

The Gaussian action in X implies that:

$$\begin{aligned}
\left\langle X^2 \right\rangle_V &= \frac{1}{DV Z_V} \frac{\partial}{\partial \omega} \int \mathcal{D}[g]_V \mathcal{D}[X_\mu]_{\rm cm}\, \mathrm{e}^{-\int \mathrm{d}^2\xi \sqrt{g}\, g^{ab}\partial_a X_\mu \partial_b X_\mu + \omega \int \mathrm{d}^2\xi \sqrt{g} X_\mu^2(\xi)} \Big|_{\omega=0} \\
&= \frac{1}{DV Z_V} \frac{\partial}{\partial \omega} \int \mathcal{D}[g]_V \Big(\det{}'(-\Delta_g - \omega)\Big)^{-D/2} \Big|_{\omega=0} \\
&= \frac{1}{2V Z_V} \int \mathcal{D}[g]_V \Big(\det{}'(-\Delta_g)\Big)^{-D/2}\, \mathrm{Tr}\,' \left[\frac{1}{-\Delta_g}\right] \\
&= \frac{1}{2V} \left\langle \mathrm{Tr}\,' \left[\frac{1}{-\Delta_g}\right] \right\rangle_V ,
\end{aligned} \tag{2.4}$$

where the primes on the determinants and traces again mean that zero modes are excluded. Formula (2.4) is used to define $\langle X^2 \rangle_V$ when D is non-integer.

Using (1.1) and (1.5) we get

$$\left\langle X^2 \right\rangle_V = \frac{1}{2} \int_0^\infty dT \, \frac{1}{T^{d_s/2}} F\Big(\frac{T}{V^{2/d_s}}\Big) \sim V^{2/d_s - 1}, \tag{2.5}$$

for V going to infinity. From (2.2) we now conclude that

$$\frac{1}{d_s} = \frac{1}{D_H} + \frac{1}{2}. \tag{2.6}$$

Several remarks are in order. Strictly speaking the above derivation assumes that $d_s < 2$. If $d_s > 2$ we have to introduce a small-T cut-off ε in (2.5). In this case it is convenient to consider instead $\langle (X^2)^n \rangle_V \sim V^{2n/D_H}$ where $n = [d_s/2] + 1$ for non-integer d_s. It is then easy to show that the leading large V behavior on the right hand side of the equation corresponding to (2.5) will be $V^{2n/d_s - 1}$ and we get

$$\frac{1}{d_s} = \frac{1}{D_H} + \frac{1}{2n} \quad \text{for} \quad 2n - 2 < d_s < 2n \;\; (\, n = 1, 2, \dots \,). \tag{2.7}$$

In the study of diffusion on fixed fractal structures one usually encounters $d_s < 2$. In the following we will *assume* $d_s \leq 2$ due to the following reasoning. We expect that $d_s \to 2$ for $D \to -\infty$ since large negative D implies that a saddle point calculation of (2.3) around a fixed geometry should be reliable. A strictly fixed geometry implies $d_s = 2$ and $D_H = \infty$ (in agreement with (2.6)). Also the saddle point calculation results in $D_H = \infty$ and should be valid in a neighborhood of $D = -\infty$. Hence $d_s = 2$ in a neighborhood of $D = -\infty$. If doing anything, one would expect fluctuating geometries to decrease D_H since there are many "degenerate" geometries where $D_H < \infty$, e.g. the branched polymer-like geometries to be discussed later. Thus it is reasonable to assume $d_s \leq 2$.

Once this assumption is made, it follows immediately that $d_s = 2$ for all $D \leq 1$ since it is known from Liouville theory that $D_H = \infty$. Let us just recall the argument[2] [10, 11]. Define the two-point function in random surface theory by:

$$G(p) = \Big\langle \iint \mathrm{d}^2\xi_1 \sqrt{g(\xi_1)} \, \mathrm{d}^2\xi_2 \sqrt{g(\xi_2)} \, \mathrm{e}^{ip(X(\xi_1) - X(\xi_2))} \Big\rangle_V . \tag{2.8}$$

If we use the following definition of $\langle X^2 \rangle_V$ (which is equivalent to (2.3) for large V)

$$\left\langle X^2 \right\rangle_V = \frac{1}{DV^2} \Big\langle \iint \mathrm{d}^2\xi_1 \sqrt{g(\xi_1)} \, \mathrm{d}^2\xi_2 \sqrt{g(\xi_2)} \big(X(\xi_1) - X(\xi_2) \big)^2 \Big\rangle_V , \tag{2.9}$$

it follows that

$$\left\langle X^2 \right\rangle_V = - \frac{1}{DV^2} \frac{\partial^2}{\partial p^2} G(p) \Big|_{p=0}. \tag{2.10}$$

Since it is known that $G(p)$ behaves as $V^{2 - \Delta_0(p)}$ in flat space with $\Delta_0(p) \propto p^2$, the KPZ formula allows us to calculate $\Delta(p)$ after coupling to gravity:

$$\Delta(p) = \frac{\sqrt{1 - D + 24\Delta_0(p)} - \sqrt{1 - D}}{\sqrt{25 - D} - \sqrt{1 - D}}. \tag{2.11}$$

[2]The treatment in [10] is based on a more general scaling assumption than is needed in two-dimensional quantum gravity and this leaves open the possibility of a scaling different from the one given here. In [11] the treatment was narrowed down to the one presented here.

It follows that for $D < 1$ we have

$$\left\langle X^2 \right\rangle_V \sim \log V \,, \tag{2.12}$$

while for $D = 1$

$$\left\langle X^2 \right\rangle_V \sim \log^2 V \,. \tag{2.13}$$

In both cases $D_H = \infty$ and thus $d_s = 2$ from (2.6).

3 Discussion

We have shown under mild assumptions that the spectral dimension $d_s = 2$ for two-dimensional quantum gravity coupled to D Gaussian fields. If we assume that the spectral dimension is a function only of the central charge of the matter fields coupled to two-dimensional quantum gravity, it follows that d_s is always two. This assumption is corroborated by numerical simulations for pure gravity, the Ising model ($c = 1/2$) coupled to gravity and the three-states Potts model ($c = 4/5$) coupled to gravity [12], as well as high-statistics simulations for $c = -2$ [13]. From the point of view of fractal structures the situation is most remarkable: the generic manifold is highly fractal when defined in the conventional way, using volume $V(R)$ versus geodesic distance R as a measure of the fractal dimension for small R. Accordingly the gap exponent δ for diffusion is large and anomalous compared to the generic value $\delta = 2$ for a smooth manifold. However, δ is exactly equal to the anomalus fractal dimension d_h and in this way the spectral dimension of the generic, fractal geometry of the two-dimensional manifold which appears in the functional integral is *two*, the same as that of a smooth, compact two-dimensional manifold!

For $D > 1$ it is generally believed that the two-dimensional surfaces degenerate to branched polymers. The Gaussian fields represent an embedding of these branched polymers into R^D and it is well known that $D_H = 4$ for the generic branched polymers. We thus conclude that the spectral dimension of branched polymers is equal to 4/3, the famous Alexander-Orbach value. The value $d_s = 4/3$ as well as formula (2.6) was derived for branched polymers by a different method in [14] (see [15] for a recent more elaborate and complete proof that $d_s = 4/3$). Furthermore it is well known that there exists a well defined class of *multicritical branched polymers* [16], very similar to the multicritical matrix models. They have $D_H = 2m/(m-1)$, $m = 2, 3, \ldots$, where the $m = 2$ class contains the ordinary branched polymers with positive branching ratios. For these we obtain $d_s = 2m/(2m-1)$. It is seen that $d_s \to 1$ for $m \to \infty$. This is in agreement with the fact that these multicritical branched polymers approach ordinary random walks for $m \to \infty$. Clearly, for an ordinary random walk where $D_H = 2$, we get $d_s = 1$ from (2.6), as expected. It is also in agreement with a very recent more elaborate analysis [17].

Finally we would like to discuss a subtlety buried in the definition (1.3) of $RP'_V(T)$. We have defined the return probability by first calculating it for a fixed geometry and then performing the functional integral over geometries. Alternatively, one could have used $K_g(\xi,\xi;T)$ to define diffusion as a function of geodesic distance R by

$$
\begin{aligned}
K'_V(R;T) &= \frac{1}{VZ_V}\int \mathcal{D}[g]_V \mathrm{e}^{-S_{\mathrm{eff}}([g])}\int \mathrm{d}\xi\sqrt{g}\cdot \\
&\cdot\int \mathrm{d}\xi'\sqrt{g'}\;\delta(d_g(\xi,\xi')-R)K'_g(\xi,\xi';T)\,, \qquad (3.1)
\end{aligned}
$$

where $d_g(\xi,\xi')$ denotes the geodesic distance from ξ to ξ' in the geometry defined by the metric g. One would be tempted to say that $RP'_V(T) = K'_V(0;T)$. However, it is not known whether the limit $R\to 0$ commutes with the functional integration. For the so-called two-point function, which is obtained from (3.1) by substituting 1 instead of K_g, it is known that the functional interal does *not* commute with limit $R\to 0$. If the limits do not commute we have two inequivalent definitions of $RP'_V(T)$. Our derivation is just based on the assumption that the expression we use satisfies (1.5). It is an interesting unsolved problem to understand if the functional integration commutes with the $R\to 0$ limit, and which definition of $RP'_V(T)$ is correct in case functional integration and $R\to 0$ are non-commuting.

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